1.

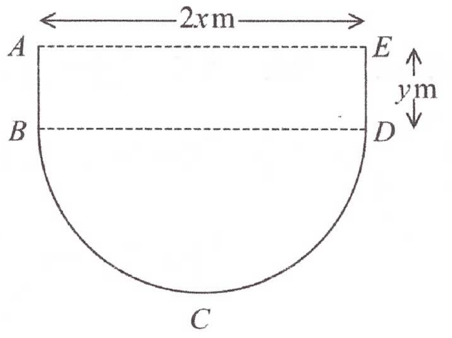


Figure 4

Figure 4 shows the plan view of the design for a swimming pool.

The shape of this pool $ABCDEA$ consists of a rectangular section $ABDE$ joined to a semicircular section BCD as shown in Figure 4.

Given that $AE = 2x$ metres, $ED = y$ metres and the area of the pool is $250\,\text{m}^2$,

(a) show that the perimeter, P metres, of the pool is given by

$$P = 2x + \frac{250}{x} + \frac{\pi x}{2}$$

(4)

(b) Explain why $0 < x < \sqrt{\frac{500}{\pi}}$

(2)

(c) Find the minimum perimeter of the pool, giving your answer to 3 significant figures.

(4)

(a) $A = 250 = 2xy + \frac{\pi(x)^2}{2}$

$250 = 2xy + \frac{\pi(x^2)}{2}$

$250 = 2xy + \frac{\pi x^2}{2}$

So $y = \frac{\left(250 - \frac{\pi x^2}{2}\right)}{2x}$

Now, $P = 2x + 2y + \frac{2\pi r}{2} = 2x + 2y + \pi r$

$= 2x + 2y + \pi x$

Substitute y as $\dfrac{\left(250-\dfrac{\pi x^2}{2}\right)}{2x}$:

$$P = 2x + \cancel{2}\dfrac{\left(250-\dfrac{\pi x^2}{2}\right)}{\cancel{2}x} + \pi x$$

$$P = 2x + \frac{250}{x} - \frac{\pi x^2}{2x} + \pi x$$

$$P = 2x + \frac{250}{x} + \pi x - \frac{\pi x^2}{2x}$$

$$P = 2x + \frac{250}{x} + \frac{2\pi x^2 - \pi x^2}{2x}$$

$$P = 2x + \frac{250}{x} + \frac{\pi x^{\cancel{2}}}{2\cancel{x}}$$

$$\therefore \boxed{P = 2x + \frac{250}{x} + \frac{\pi x}{2}}$$

(b) $x > 0$ and $y > 0$ since both are lengths

$x > 0$ and $\dfrac{\left(250-\dfrac{\pi x^2}{2}\right)}{2x} > 0$

$$250 - \frac{\pi x^2}{2} > 0$$

$$250 > \frac{\pi x^2}{2}$$

$$500 > \pi x^2$$

$$\frac{500}{\pi} > x^2$$

$$x^2 < \frac{500}{\pi}$$

$$\therefore \ x < \sqrt{\frac{500}{\pi}}$$

And together with $x > 0$, $\boxed{0 < x < \sqrt{\frac{500}{\pi}}}$

(c) $P = 2x + \frac{250}{x} + \frac{\pi x}{2}$

$$P = 2x + 250x^{-1} + \frac{\pi}{2}x$$

$$\frac{dP}{dx} = 2 - 250x^{-2} + \frac{\pi}{2}$$

$$= 2 - \frac{250}{x^2} + \frac{\pi}{2}$$

At minimum, $\frac{dP}{dx} = 0$, so $2 - \frac{250}{x^2} + \frac{\pi}{2} = 0$

$$2 + \frac{\pi}{2} = \frac{250}{x^2}$$

$$2x^2 + \frac{\pi}{2}x^2 = 250$$

$$\left(2 + \frac{\pi}{2}\right)x^2 = 250$$

$$x^2 = \frac{250}{\left(2 + \frac{\pi}{2}\right)}$$

$$x^2 = 70.012\ldots$$

$$\underline{x = 8.36\text{m (to 3s.f.)}}$$

Substitute x into $P = 2x + \frac{250}{x} + \frac{\pi x}{2}$

$$P = 2(8.36) + \frac{250}{8.36} + \frac{\pi(8.36)}{2}$$

$$P = 59.744\ldots$$

$$\boxed{P = 59.8\text{m (to 3s.f.)}}$$

2. A lorry is driven between London and Newcastle.

In a simple model, the cost of the journey £C when the lorry is driven at a steady speed of v kilometres per hour is

$$C = \frac{1500}{v} + \frac{2v}{11} + 60$$

(a) Find, according to this model,

(i) the value of v that minimises the cost of the journey,

(ii) the minimum cost of the journey.
(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(6)

(b) Prove by using $\frac{d^2C}{dv^2}$ that the cost is minimised at the speed found in (a)(i).

(2)

(c) State one limitation of this model.

(1)

ai) $\frac{dC}{dv} = 0$

$C = 1500v^{-1} + \frac{2}{11}v + 60$

$\frac{dC}{dv} = (-1)(1500v^{-2}) + \frac{2}{11}$

$= -\frac{1500}{v^2} + \frac{2}{11}$

$\frac{2}{11} - \frac{1500}{v^2} = 0$

$\frac{2}{11} = \frac{1500}{v^2}$

$2v^2 = 11(1500)$

$v = \sqrt{8250}$

$= 90.8 \text{ kmh}^{-1}$

ii) $C = \frac{1500}{90.8} + \frac{2(90.8)}{11} + 60$

$= 93.0289$

$\simeq £93.03$

b) $\frac{d^2C}{dv^2} = (-2)\frac{(-1500)}{v^3}$

$= \frac{3000}{v^3}$

$v = 90.8 \quad \frac{d^2C}{dv^2} = 0.004 > 0$

$\therefore$ minimum point

c) The speed throughout the whole journey cannot be kept constant.

3. A curve has equation $y = \mathrm{g}(x)$.

Given that

- $\mathrm{g}(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = \mathrm{g}(x)$ passes through the origin
- the curve with equation $y = \mathrm{g}(x)$ has a stationary point at (2, 9)

(a) find $\mathrm{g}(x)$,

(7)

(b) prove that the stationary point at (2, 9) is a maximum.

(2)

tick off properties as you go to keep track

a) cubic: $g(x) = ax^3 + bx^2 + cx + d$

x^3 coeff. = x coeff. $\Rightarrow g(x) = \underline{a}x^3 + bx^2 + \underline{a}x + d$

passes through origin $\Rightarrow d = 0,\ g(x) = ax^3 + bx^2 + ax$

passes through $(2,9) \Rightarrow 9 = 8a + 4b + 2a$

$\Rightarrow 10a + 4b = 9$ ①

$(2,9)$ is a stationary point $\Rightarrow g'(2) = 0$

$g'(x) = 3ax^2 + 2bx + a$

$\Rightarrow 0 = 12a + 4b + a$

$13a + 4b = 0$ ②

② − ①: $3a = -9$

$a = -3$

$\Rightarrow b = \dfrac{9 + 10(3)}{4}$

$= \dfrac{39}{4}$

so $g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$

b) for a maximum, $g''(x) < 0$

$g''(x) = 2 \times 3x - 3x + 2 \times \frac{39}{4}$

$= -18x + \frac{39}{2}$

$g''(2) = -18(2) + \frac{39}{2}$

$= -\frac{33}{2} < 0$ hence point is a max.

4.

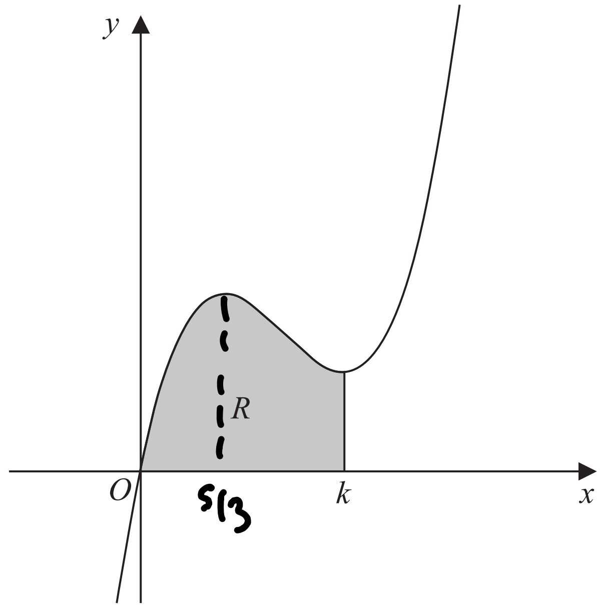


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2x^3 - 17x^2 + 40x$$

The curve has a minimum turning point at $x = k$.

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the line with equation $x = k$.

Show that the area of R is $\frac{256}{3}$

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

a) $\frac{dy}{dx} = 6x^2 - 34x + 40 = 0$

$(3x - 5)(2x - 8) = 0$

$x = \frac{5}{3}$ $\quad x = 4$

$x = \frac{5}{3}$ corresponds to the 1st turning point, so $x = 4$ is the one we are after. ie $k = 4$.

$$\text{Area}_R = \int_0^4 [2x^3 - 17x^2 + 40x]\,dx$$

$$= \left[2x^4/4 - 17x^3/3 + 20x^2\right]_0^4$$

$$= \left[\tfrac{1}{2}(256) - \tfrac{17}{3}(64) + 20(16)\right]$$

$$= \left[128 - \frac{1088}{3} + 320\right]$$

$$= \left[448 - \frac{1088}{3}\right]$$

$$= \boxed{\frac{256}{3}}$$

5. A company decides to manufacture a soft drinks can with a capacity of 500 ml.

Volume = 500ml

The company models the can in the shape of a right circular cylinder with radius r cm and height h cm.

In the model they assume that the can is made from a metal of negligible thickness.

(a) Prove that the total surface area, S cm^2, of the can is given by

$$S = 2\pi r^2 + \frac{1000}{r}$$

(3)

Given that r can vary,

(b) find the dimensions of a can that has minimum surface area.

(5)

(c) With reference to the shape of the can, suggest a reason why the company may choose not to manufacture a can with minimum surface area.

(1)

a) $SA = 2\pi r^2 + 2\pi r h$

$V = \pi r^2 h$

$\pi r^2 h = 500$ – 1

$h = \frac{500}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r \times \left(\frac{500}{\pi r^2}\right)$ – 1

$S = 2\pi r^2 + \frac{1000\cancel{\pi r}}{\cancel{\pi r^2}}$

$S = 2\pi r^2 + \frac{1000}{r}$ – 1

b) $S = 2\pi r^2 + 1000 r^{-1}$

$\frac{dS}{dr} = 4\pi r - 1000 r^{-2}$

$= 4\pi r - \frac{1000}{r^2}$ – 2

@ $\frac{dS}{dr} = 0$ – 1

$4\pi r - \frac{1000}{r^2} = 0$

$4\pi r^3 - 1000 = 0$

$r^3 = \frac{1000}{4\pi}$

b) $r = 4.30\text{ cm}$ (3 s.f.) - 1

$h = \frac{500}{\pi r^2}$

$h = \frac{500}{\pi(4.30)^2}$

$h = 8.60\text{cm}$ (3s.f.) - 1

Radius = 4.30 cm (3s.f.)
Height = 8.60 cm (3s.f.)

c) $r = 4.30\text{cm}$ $h = 8.60\text{ cm}$

If the radius is 4.30cm and the height is 8.60cm, then the can is square in profile - but all cans are taller than they are wide - 1

6. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \qquad x > 0$$

(a) Find (i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 4$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

a) i) $y = x^2 - 2x - 24x^{1/2}$

$y = ax^n, \ \dfrac{dy}{dx} = anx^{n-1}$

$\dfrac{dy}{dx} = 2x - 2 - 12x^{-1/2}$ ✓✓

ii) $\dfrac{d^2y}{dx^2} = 2 + 6x^{-3/2}$ ✓

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

b) $\dfrac{dy}{dx} = 2x - 2 - 12x^{-1/2}$

S.p. is when $\left.\dfrac{dy}{dx}\right|_{x=a} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=4} = 2(4) - 2 - 12(4)^{-1/2}$ ✓

$= 8 - 2 - 6 = 0$ ✓

$\left.\dfrac{dy}{dx}\right|_{x=4} = 0$ ✓, hence a stationary point at $x = 4$. ✓

c) $\frac{d^2y}{dx^2} = 2 + 6x^{-3/2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=a} > 0 \rightarrow$ minimum

$\left.\frac{d^2y}{dx^2}\right|_{x=b} < 0 \rightarrow$ maximum

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = 2 + 6(4)^{-3/2}$$

$$= 2 + \frac{6}{8} = 2.75$$

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = 2.75 > 0$, hence stationary point is a minimum

7.

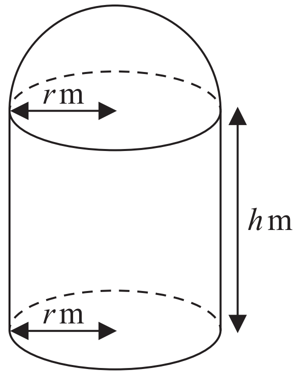


Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is $6\,\text{m}^3$.

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)

a) $A = A_1 + A_2 + A_3$

Area Cylinder : $A_1 = 2\pi r h$

Area base : $A_2 = \pi r^2$

Area hemisphere : $A_3 = 2\pi r^2$

$V = 6\text{m}^3 = V_{cylinder} + V_{hsphere}$

$6 = \pi r^2 h + \frac{2\pi r^3}{3}$ 1 $\Rightarrow \pi r^2\left(h + \frac{2}{3}r\right) = 6$

$\Rightarrow h = \frac{6}{\pi r^2} - \frac{2}{3}r$ 1

$\Rightarrow A = 2\pi r h + \pi r^2 + 2\pi r^2$ 1

$\Rightarrow A = 2\pi r h + 3\pi r^2$

$\Rightarrow A = \left(\frac{6}{\pi r^2} - \frac{2}{3}r\right)2\pi r + 3\pi r^2$

$\Rightarrow A = \frac{12\pi r}{\pi r^2} - \frac{4}{3}\pi r^2 + 3\pi r^2$

$\Rightarrow A = \frac{12}{r} + \frac{5}{3}\pi r^2$ 1 as required.

b) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$

① Differentiate
② Set equal 0
③ Solve for r

$\frac{dA}{dr} = -\frac{12}{r^2} + \frac{10\pi r}{3} = 0$ ②

$\Rightarrow -\frac{36}{r^2} + 10\pi r = 0$

$\Rightarrow -36 + 10\pi r^3 = 0 \Rightarrow 10\pi r^3 = 36$

$\Rightarrow r^3 = \frac{36}{10\pi}$ ① $\Rightarrow r = \sqrt[3]{\frac{36}{10\pi}} = 1.05\text{m}$

$\Rightarrow$ Surface area will be a minimum when $r = 1.05\text{m}$ ①

c) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$, $r = 1.05\text{m}$

$A = \frac{12}{1.05} + \frac{5}{3}\pi(1.05)^2 = 17.201\ldots$ ①

$\Rightarrow$ Minimum Surface is 17m^2 ①

8. Given that

$$f(x) = x^2 - 4x + 5 \qquad x \in \mathbb{R}$$

(a) express f(x) in the form $(x + a)^2 + b$ where a and b are integers to be found.

(2)

The curve with equation $y = f(x)$

- meets the y-axis at the point P
- has a minimum turning point at the point Q

(b) Write down

(i) the coordinates of P

(ii) the coordinates of Q

(2)

(a) $f(x) = x^2 - 4x + 5$

$= \left(x + \frac{(-4)}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 + 5$

$= (x-2)^2 - 4 + 5$

$= (x-2)^2 + 1$ ① ①

$\therefore a = -2, b = 1$

(b)(i) meets y-axis means $x = 0$

when $x = 0$, $f(0) = (0-2)^2 + 1$

$= 4 + 1$

$= 5$

$\therefore P(0, 5)$ ①

(ii) minimum turning point can be directly found by looking at $f(x) = (x-2)^2 + 1$

$\therefore Q = (2, 1)$ ①

9.

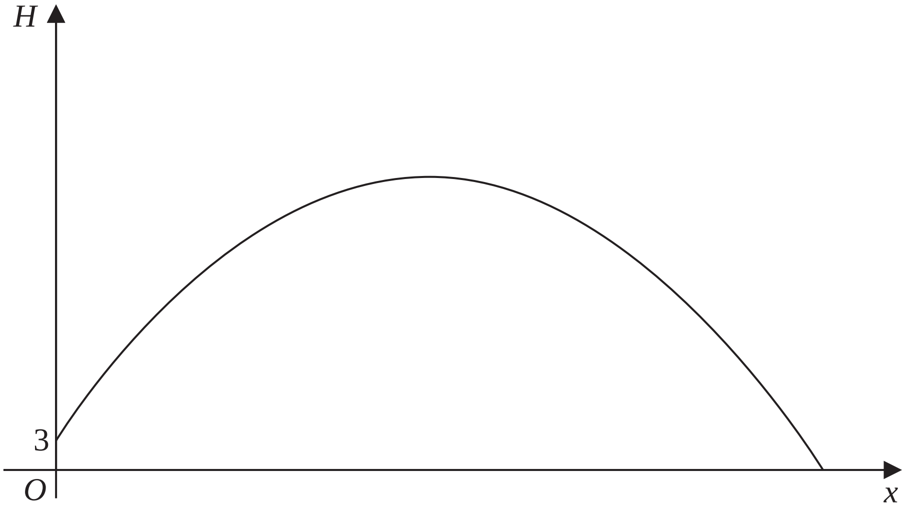


Figure 3

Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, H metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 3 m above the ground
- reaches its maximum vertical height after travelling a horizontal distance of 90 m
- is at a vertical height of 27 m above the ground after travelling a horizontal distance of 120 m

Given also that H is modelled as a **quadratic** function in x

(a) find H in terms of x

(5)

(b) Hence find, according to the model,

(i) the maximum vertical height of the ball above the ground,

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

(3)

(c) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model.

(1)

DO NOT WRITE IN THIS AREA

(a) $H = ax^2 + bx + c$

$x = 0,\ H = 3:\quad 3 = 0 + 0 + c$

$c = 3$

$H = ax^2 + bx + 3$ — (1)

$x = 120,\ H = 27:\quad 27 = a(120)^2 + b(120) + 3$

$27 = 14400a + 120b + 3$ — (1)

$24 = 14400a + 120b$ — (1)

$\frac{dH}{dx} = 2ax + b$ — (1)

$x = 90,\ \frac{dH}{dx} = 0:\quad 0 = 2a(90) + b$

$= 180a + b$ — (1)

$b = -180a$ — (2)

Substitute (2) into (1)

$24 = 14400a + 120(-180a)$

$= 14400a - 21600a$

$= -7200a$

$a = -\frac{1}{300}$

$b = -180\left(-\frac{1}{300}\right)$

$= \frac{3}{5}$

$\therefore H = -\frac{x^2}{300} + \frac{3x}{5} + 3$ # (1)

(b)(i) $x = 90$: $H = \frac{-(90)^2}{300} + \frac{3(90)}{5} + 3$

$= 30 \text{ m}$ (1)

(ii) $H = 0$: $0 = \frac{-x^2}{300} + \frac{3x}{5} + 3$ (1)

$x = 184.86\ldots, -4.868\ldots$

$\therefore x = 185 \text{ m}$ (nearest metre) (1)

only take positive value as the answer

(c) The ground is unlikely to be horizontal (1)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

10. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) i) $\dfrac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ ① ①

use rule $ax^n \rightarrow anx^{n-1}$ to differentiate x

e.g. $5x^4 \rightarrow 5 \times 4 \times x^{4-1} = 20x^3$

ii) $\dfrac{d^2y}{dx^2} = 60x^2 - 144x + 84$ ①

b) i) when $x = 1$, gradient $\dfrac{dy}{dx} = 20(1^3) - 72(1^2) + 84(1) - 32$ ①

$= 20 - 72 + 84 - 32$

$= 0$

$\therefore \dfrac{dy}{dx} = 0$ so there is a stationary point at $x = 1$. ①

ii) when $x = 0.8$, $\dfrac{dy}{dx} = 20(0.8^3) - 72(0.8^2) + 84(0.8) - 32$ ← choose x values either side of the stationary point.

$= -0.64$

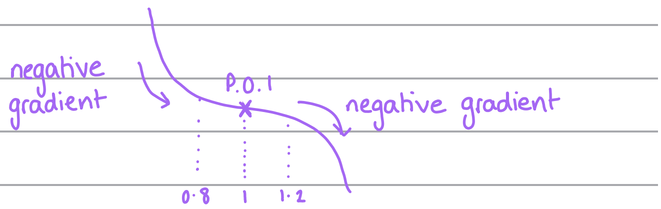
$-0.64 < 0$ so gradient is negative

when $x = 1.2$, $\dfrac{dy}{dx} = 20(1.2^3) - 72(1.2^2) + 84(1.2) - 32$

$= -0.32$

$-0.32 < 0$ so gradient is negative. ① for finding both gradients

Since the gradient is negative on both sides of the stationary point, this must be a point of inflection. ①



P 6 8 7 3 2 A 0 2 0 4 8

11.

The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

a)

i) $y = 3x^4 - 8x^3 - 3 \Rightarrow \frac{dy}{dx} = \underline{\underline{12x^3 - 24x^2}}$ 1 1

ii) $\frac{d^2y}{dx^2} = \underline{\underline{36x^2 - 48x}}$ 1

b) Stationary point when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 12x^3 - 24x^2 \Rightarrow 12(2)^3 - 24(2)^2 = 12 \times 8 - 24 \times 4 = 0$ 1

$\Rightarrow$ At $x = 2$, $\frac{dy}{dx} = 0 \Rightarrow x = 2$ is a stationary point. 1

c) $\frac{d^2y}{dx^2}$ and substitute in $x = 2$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ Minimum

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ Maximum

$\left.\frac{d^2y}{dx^2}\right|_2 = 36(2)^2 - 48(2) = 144 - 96 = 48$ 1

$\Rightarrow 48 > 0 \Rightarrow$ Stationary point which is a $\underline{\underline{\text{minimum}}}$. 1